Circular Motion and Moment of Inertia

Saba Karakas

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1 Theoretical Background

1.1 Angular Momentum and Torque

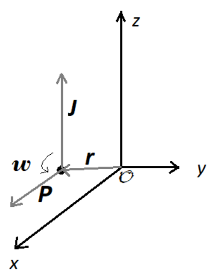

Figure 1: Linear momentum, position and angular momentum vectors of a point particle rotating about z-axis.

Angular momentum is the rotational equivalent of linear momentum in circular motions. The angular momentum of a point particle about a fixed axis is defined as,

$$\vec{J} = \vec{r} \times \vec{P}, \tag{1}$$

where $\vec{r}$ and $\vec{P}$ is the position and the linear momentum vectors, respectively. A visual representation may be found in Figure (1). If the force $\vec{F}$ acts on the particle, then we define the *torque* or *moment of force* as follows,

$$\vec{\tau} = \vec{r} \times \vec{F}. \tag{2}$$

To add another definition for the torque, now we may differentiate the angular momentum with respect to time and this yields,

$$\begin{aligned} \frac{\partial \vec{J}}{\partial t} &= \underbrace{\frac{\partial \vec{r}}{\partial t} \times \vec{P}}_{\vec{V}\times(m\vec{V})=0} + \vec{r} \times \frac{\partial \vec{P}}{\partial t}, \\ &= \vec{r} \times \frac{\partial \vec{P}}{\partial t}. \end{aligned} \tag{3}$$

Here we may recall the definition of force according to Newton's 2nd law of motion as,

$$\vec{F} = \frac{\partial \vec{P}}{\partial t}. \tag{4}$$

Evaluating Eqn (3) and Eqn (4) together yields another definition for the **torque**,

$$\vec{\tau} = \frac{\partial \vec{J}}{\partial t}. \tag{5}$$

To this respect, one may define the torque as the time rate of change of angular momentum.

After defining the basic concepts around a rotating point particle, now let's consider a rigid body rotating about an axis that passes through it. Thus the constituent elements of the body on that axis remain stationary, i.e. have no angular velocity. In other words, they remain still on a fixed line in space. Since we know somethings about the rotational dynamics about a point particle, it is reasonable to choose a very small portion of the rigid body and examine its dynamics to investigate the motion further. The chosen portion, i.e. a constituent element of the body, is shown in Figure (2) labeled with point-P. I choose the projection point of the constituent element on the rotation axis to be the origin of my coordinate system. This way I assure the perpendicularity of the rotation axis and the position vector (see Figure (2)). This will simplify the calculations a little bit. And also I describe the rotation by its angular velocity $\vec{\omega}$. Let's focus the constituent element of the rigid body at point-P of mass m. The instantaneous velocity and acceleration of this element is,

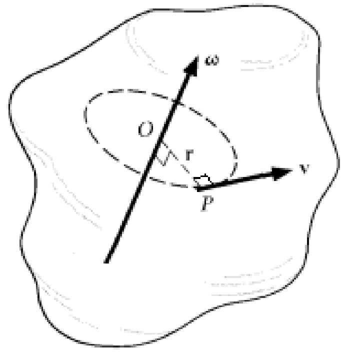

Figure 2: At the moment pictured, rotation of the body causes point p to move in a circle of radius r in a plane perpendicular to $\vec{w}$. The magnitude of $\vec{v}$ is $v = wr \sin 90° = wr$. and its direction is normal to the plane defined by $\vec{w}$ and $\vec{r}$. Thus $\vec{v} = \vec{w} \times \vec{r}$.

$$\vec{v} = \vec{\omega} \times \vec{r}, \tag{6}$$
$$\vec{a} = \vec{\alpha} \times \vec{r}, \tag{7}$$

where $\vec{\alpha}$ is the angular acceleration of it. Indeed, this element contributes some angular momentum to the total angular momentum of the body. The contribution of this element is $\vec{r} \times m\vec{v} = \vec{r} \times m(\vec{\omega} \times \vec{r})$. Recognize that the body has many other constituent elements just like we considered. Therefore, the total angular momentum of the body may be written as the summation of the angular momentum contributions of all of these constituent elements.

$$\vec{J} = \sum_i \vec{r_i} \times m_i(\vec{\omega} \times \vec{r_i})^1, \tag{8}$$

where the index, i represents the individual constituent elements. The magnitude of the angular momentum results in the definition of "***the moment of inertia***".

$$J = \underbrace{\sum_i m_i r_i^2}_{\substack{\textit{moment of} \\ \textit{inertia}}} \omega. \tag{9}$$

A quick look in Figure (2) makes you realize that both of the angles required in vectorial multiplications in Eqn (8) are 90° and this is the reason we can get rid of the sinus elements originated by the vectorial multiplications. On the other hand, it is conventional to label the rotation axis as the z-axis of the reference frame. Therefore the moment of inertia about z-axis is,

$$I_z = \sum_i m_i r_i^2. \tag{10}$$

If we would choose infinitesimal constituent elements, their mass would be dm and the definition given with Eqn (10) yields an integral and turns into,

$$I_z = \int r^2 dm. \tag{11}$$

[1] I need to highlight that the angular velocity of the constituent elements of a rigid body <u>is</u> the same, of course. Otherwise the body would break into pieces due to the rotation.

Now we may rewrite the definitions of both the angular momentum and the torque by using the moment of inertia, as follows.

$$J = I\omega, \tag{12}$$
$$\tau = I\alpha. \tag{13}$$

where α is the angular acceleration. In the experiment, we will use a cylindrical rigid body. For this reason, the moment of inertia of a cylindrical rigid body will be evaluated as an example.

Example: Moment of Inertia of a Cylinder

Let's consider a homogeneous rigid cylindrical body rotating around an axis which is passing through its center and is perpendicular to both of its circular caps (see Figure (3)). For a cylinder, it is obviously more convenient to choose the cylindrical coordinate system. To use Eqn (11), we need an infinitesimal mass element, dm. We may express this mass element, dm, in terms of the density and volume element. Volume element in cylindrical coordinates is,

$$dV = rdrd\phi dz, \tag{14}$$

where r, ϕ and z are the cylindrical coordinates. The definition of them is shown in Figure (3). To learn more about cylindrical coordinates and how come we end up this volume element, you may check Ref[2].

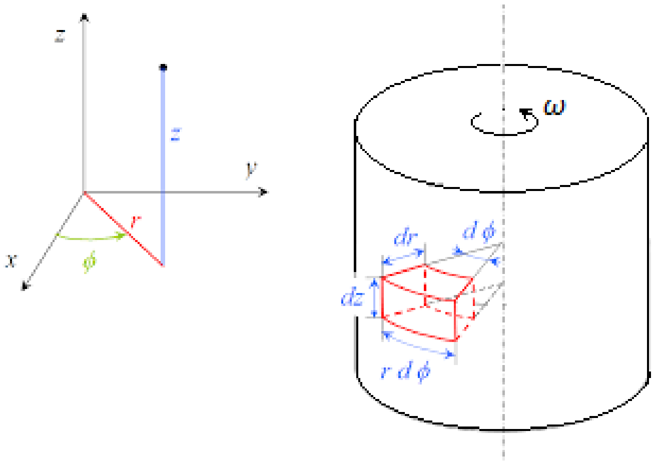

Figure 3: Definition of cylindrical coordinates and an exaggerated sketch for volume element in cylindrical coordinates. Rigid cylindrical body is rotating around z-axis.

By using the density, one can write the mass element in terms of the volume element in any coordinate system. For cylindrical coordinates, this formula is as follows,

$$\begin{aligned} mass &= density \times volume\ element \\ dm &= \rho rdrd\phi dz. \end{aligned} \tag{15}$$

where ρ is the density of the cylinder and it is,

$$\rho = \frac{M}{\pi R^2 L}, \tag{16}$$

where M is the total mass, R is the radius and L is the length of the cylinder. By using Eqn (11), Eqn (15) and Eqn (16), we may obtain the moment of inertial of a cylinder as follows:

$$\begin{aligned} I_z &= \int r^2 dm \\ &= \int_{r=0}^{R}\int_{\phi=0}^{2\pi}\int_{z=0}^{L} r^2 \underbrace{\frac{M}{\pi R^2 L}}_{const.} rdrd\phi dz \\ &= \frac{M}{\pi R^2 L}\int_{r=0}^{R} r^3 dr \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{L} dz \\ &= \frac{M}{\pi R^2 L}\left.\frac{r^4}{4}\right|_{r=0}^{R} \phi|_{\phi=0}^{2\pi}\, z|_{z=0}^{L} \\ &= \frac{1}{2}MR^2 \end{aligned} \tag{17}$$

If you wonder how we decide the limits of integrals, you should check Ref.[2], again.

Example: System in The Experiment

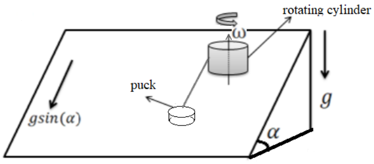

Figure 4: System of a rigid cylinder rotated by the puck moving downwards on the airtable.

The system we'll be investigating in the experiment consists of two main elements. One of them is, of course, a puck moving on the frictionless inclined plane. The other is a rotating rigid cylinder without any friction. The puck and the cylinder is attached to one another with an inelastic rope which is reeled up on the cylinder initially. You may see a sketch of the system in Figure (4). Free body diagram of the puck and the cylinder is shown in Figure (5). We'll deal with them one by one. First, we consider the cylinder. The magnitude of the torque on the cylinder due to the tension in the rope is,

$$\tau = RT, \tag{18}$$

and the magnitude of the angular acceleration of the cylinder is,

$$\alpha = \frac{a}{R}. \tag{19}$$

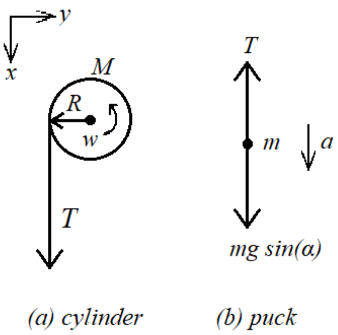

Figure 5: Free body diagram for the experimental setup.

By using these two equations and Eqn (13) together, we get,

$$RT = I_z \frac{a}{R}. \tag{20}$$

We have found the moment of inertia of a cylinder rotating around an axis passing through its center perpendicular to its circular capes in Eqn (17). By using that formula and the last equation above, we get,

$$T = \frac{1}{2} Ma. \tag{21}$$

On the other hand, for puck the equation of motion is as follows,

$$mg \sin\alpha - T = ma. \tag{22}$$

Please not that, here the α is the inclined plane angle, not the angular acceleration. By using Eqn (21), we can evaluate the acceleration of the system, as follows,

$$a = \frac{m}{m + \frac{M}{2}} g \sin\alpha. \tag{23}$$

2 Procedure

2.1 Experimental Procedure

1. Turn the airtable into an inclined plane by placing its cylindirical part under its back foot.
2. Turn on the lab table and then the airtable.
3. By using only compressor's pedal, make sure the airtable is working.
4. If the airtable is working; turn it off. If not; contact with your lab instructor.
5. Set the frequency of the airtable to $10Hz$ or $20Hz$ and write this value down in Section 3, Eqn (27).
6. Attach the cylindrical system on the top of the airtable.
7. Place the carbon paper onto the airtable.
8. Place the experiment sheet onto the carbon paper.
9. Arrange the 1st mass value for puck given in Table (1).
10. Hook the ring at the end of the rope of the cylinder to the puck and reel up the rope around the gutter of the cylinder clockwise.
11. Turn on the airtable and by pushing both of the pedals, let the puck move downwards. The movement of the puck should make the cylinder rotate. Keep pushing the pedals until all of the rope is unwind.
12. Take the experiment sheet off of the airtable and label it with "#1".
13. Repeat the steps 8-12 with other mass values on Table (1), label the experiment sheets with increasing numbers.

2.2 Analysis Procedure

2.2.1 Prep work

- Measure the mass of the cylinder and note it on Eqn (29).
- By using a vernier caliper measure the radius of the cylinder. Write it down to Eqn (30).
- By using a vernier caliper measure the radius of the gutter of the cylinder and record it to Eqn (31)[2].
- Measure the mass of bare puck and write in in Eqn (32).

2.2.2 How to fill the tables

1. By examining all of the experiment sheets, decide a total interval number, n and write it down in Eqn (28).
2. Take the experiment sheet #1. Label the experimental dots beginning with the very first one as 0, continue with the labeling until you reach the n_th dot.
3. Measure the total distance between 0th and n_th dots and write it down in Table (1).

[2]If you take a closer look to the cylinder, you'll see that there is a gutter on it and the rope is reeled up here. Therefore the tension in the rope acts on the cylinder not from R but from a closer point. We call this point as "the radius of the gutter".

4. Using the formula,

$$t_n = \frac{n}{f}, \tag{24}$$

calculate the total time elapsed and note it in Table (1).

5. By using the distance-time formula for the motion with acceleration with vanishing initial position and velocity,

$$x(t) = \frac{1}{2}at^2, \tag{25}$$

calculate the experimental acceleration value and fill the related cells on Table (1) and Table (1).

6. By using the formula Eqn (19), calculate the experimental and the theoretical angular accelerations and write them down on Table (1).

7. By using the percentage error formula,

$$P.E. = \% \frac{|a_{the} - a_{exp}|}{a_{the}} \times 100 \tag{26}$$

calculate the percentage error in the angular acceleration and write it in Table (1).

8. By using Eqn (21), calculate the tension in the rope and write it in Table (1).

9. By using Eqn (13), calculate the torque value and fill the cell in Table (1).

10. Repeat the steps 2-9 with other experiment sheets.

2.2.3 Plots & Analysis of data

We'll plot the graph of α-τ to investigate the moment of inertia. Because if one pays attention to Eqn (13), the slope of α-τ graph should yield the moment of inertia (see Ref[1] for more info).

1. Plot the graph of α_{exp}-τ.

2. Calculate the slope of it. This value is your experimental value for moment of inertia of a cylindrical object.

3. By using Eqn (17), calculate the theoretical value for moment of inertia of a cylindrical object.

4. By using the percentage error formula, calculate the percentage error in moment of inertia.

3 Data & Results

- The frequency:

$$f = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}Hz. \tag{27}$$

- Total number of intervals:

$$n = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}. \tag{28}$$

- Mass of the cylinder:

$$M = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}gr. \tag{29}$$

- Radius of the cylinder:

$$R = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}cm. \tag{30}$$

- Inner radius of the gutter of the cylinder:

$$r = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}cm. \tag{31}$$

- Bare mass of the puck:

$$m_{puck}^{bare} = \text{\_\_\_\_\_\_\_\_\_\_\_\_}gr. \tag{32}$$

- Measurements for a_{exp}:

- Measurements for α_{exp}, :

- Calculations for a_{the}:

- Calculations for α_{the}:

- Calculations for P.E. of α:

- Calculations for T:

- Calculations for τ:

- Calculations for I_z^{exp}:

- Calculations for I_z^{the}:

- Calculations for P.E. of I_z:

#	$m\ (g)$	$x\ (cm)$	$t\ (s)$	$a_{exp}\ (cm/s^2)$	$\alpha_{exp}\ (s^{-2})$	$a_{the}\ (cm/s^2)$	$\alpha_{the}\ (s^{-2})$	P.E.	$T\ (dyn)$	$\tau\ (dyn\ cm)$
1	575							%		
2	600							%		
3	625							%		
4	650							%		
5	675							%		

Table 1:

#	a_{exp} (cm/s^2)	α (rad/s^2)	T (dyn)	τ $(dyn\ cm)$
1				
2				
3				
4				
5				

Table 2: Data to plot the α_{exp} - τ graph.

4 Conclusions

5 Notes

References

[1] Saba Arife Karakaş. Graphics guide. online published, “fzk.fef.marmara.edu.tr”, pg6, 10/2018.

[2] Wikipedia contributors. Cylindrical coordinate system — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Cylindrical_coordinate_system&oldid=866436589, 2018. [Online; accessed 17-November-2018].