Simple Pendulum

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1 Theoretical Background

A simple pendulum is a system consist of a mass attached to one end of a massless rope whose other end is attached to a wall. A simple pendulum can be seen in Figure(1a) If one pulls the mass a little bit and release, then the mass starts to swing. The arc length taken by the mass is during the swing is,

$$s = L\theta, \tag{1}$$

where L and θ are length of the rope and the angle between the rope and the vertical axis. The speed and the magnitude of the acceleration is as follows,

$$v = \frac{ds}{dt} = \frac{d}{dt}(L\theta) = L\frac{d\theta}{dt} = L\omega, \tag{2}$$
$$a = \frac{d^2s}{dt^2} = \frac{d^2}{dt^2}(L\theta) = L\frac{d^2\theta}{dt^2} = L\alpha, \tag{3}$$

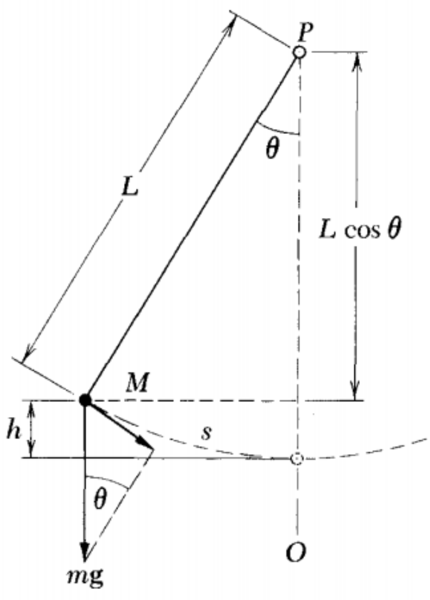

Figure 1: A simple pendulum[1].

where v and a are the magnitudes of instantaneous velocity and instantaneous acceleration, respectively. Please do remember that the angular velocity is first order time derivative of the angle and the angular acceleration is the second order time derivative of it. There are two forces in this system: the force of gravity and the tension in the rope. If we use the polar coordinates[2], the tension in the rope would be in the $\hat{r}$ direction and this force is balanced by the $\hat{r}$ component of the weight of the mass. Thus there is no motion in this direction. But in the other direction, there is $mgsin(\theta)$ component of the weight. This component is not balanced and cause the motion reduces the θ, initially.

$$mgsin(\theta) = -m\frac{d^2s}{dt^2}. \tag{4}$$

The minus sign in this equation is originated by the decrease in θ. On the other hand, the series expansion for $sin(\theta)$ is,

$$sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \tag{5}$$

And for small θ values we may neglect the elements of $O(\theta^2)$. This is called "small angle approximation"[3]. Thus we get,

$$\begin{aligned} mgsin(\theta) &\approx mg\theta, \\ mg\theta &= -m\ddot{s}. \end{aligned} \tag{6}$$

Here we should remember that, $\ddot{s} = L\ddot{\theta}$. Then we get,

$$mg\theta = -mL\ddot{\theta}. \tag{7}$$

Let's put this equation in order and rewrite it as,

$$\ddot{\theta} + \frac{g}{L}\theta = 0. \tag{8}$$

Eqn (8) is a 2nd order differential equation with constant coefficients. These types of equations also known as "wave equation" and their solutions are trigonometric functions. We may propose a solution such as,

$$\theta(t) = A\ cos(\omega t + \phi). \tag{9}$$

Here ω is a constant of dimension $Time^{-1}$ and ϕ is a dimensionless constant. The physical meaning of these constants will be discussed in a moment. Now, if we derivate this proposal with respect to time twice,

$$\ddot{\theta} = -\omega^2 A\ cos(\omega t + \phi), \tag{10}$$

and substitute them into Eqn (8), we get,

$$\begin{aligned} \underbrace{\ddot{\theta} + \frac{g}{L}\theta}_{\Downarrow} &= 0 \\ -\omega^2 A\ cos(\omega t + \phi) + \frac{g}{L} A\ cos(\omega t + \phi) &= 0 \\ \left(-\omega^2 + \frac{g}{L}\right) A\ cos(\omega t + \phi) &= 0 \end{aligned} \tag{11}$$

The general solution of this equation is,

$$\omega = \sqrt{\frac{g}{L}}. \tag{12}$$

This solution means that if *omega* is equal to the square root of gravitational acceleration over length of rope, then the solution proposed in Eqn (9) is the solution of the differential equation given in Eqn (8). Please remember that, ω is a constant of dimension $1/Time$. In physics, these kind of constants are of dimension of frequency. And indeed ω is the angular frequency of this system. It defines how frequently this motion repeats itself by means of the swiped angle. On the other hand, we said that ϕ is a dimesionless coefficient. And this coefficient is called as "*phase*". It defines how far away the system begins to oscillate from its maximum angular amplitude, θ_0. Now we may rewrite the solution in the light of this information.

$$\theta(t) = A\ cos\left(\sqrt{\frac{g}{L}}t + \phi\right). \tag{13}$$

In order to find A, we need initial conditions. Let's name the initial angle as θ_0 and we take the phase as zero. Then,

$$\theta(0) = \theta_0 = A\ \underbrace{cos(\omega 0 + 0)}_{=1} \quad \Rightarrow \quad A = \theta_0 \tag{14}$$

Let's rewrite the solution last time, taking into account all we know, as follows,

$$\theta(t) = \theta_0\ cos\left(\sqrt{\frac{g}{L}}t\right). \tag{15}$$

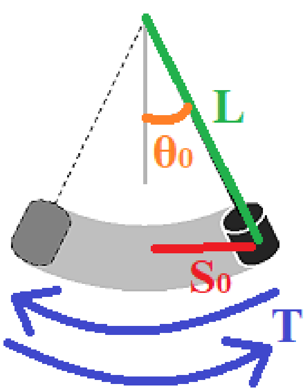

Figure 2: Pictorial definitions of "*rope length*" (L), "*angular amplitude*" (θ_0), "*amplitude*" (s_0) and "period" (T). T is of course the time spent during the motion described with blue arrows. θ_0 is an angle. L and s_0 are lengths.

Another significant factor in this motion is the frequency of the system and is defined as,

$$f = \frac{\omega}{2\pi}, \tag{16}$$

in terms of the angular frequency. Then it is,

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}. \tag{17}$$

In addition to this, the period of an oscillation is inverse of its frequency,

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}}. \tag{18}$$

In this experiment we will investigate the transition of this period with the length of the rope.

2 Procedure

2.1 Experimental Procedure

This experiment will be performed within 3 parts. And every measurement in every part will be repeated three times, then we'll calculate their mean value in analysis part. Also, we'll measure not the period itself but 10 times the period just so the error in a single measurement decreases.

2.1.1 Part A: Effect of mass difference

In this part, the rope length and the amplitude will remain same. We'll only change the mass of the body. What is your expectation?

1. Measure the mass values of two different pendulum bodies and write them in Table (1).

2. There is a long rope with several knots, hanged to the ceiling in our lab. Select 1 of these knots and measure the rope length. Write the length value in Eqn (21).

3. Choose an amplitude and measure it. Write this value in Eqn (22).

4. Hang the 1st body at the knot.

5. Make your timer ready.

6. Pull the body to the amplitude you've choosed for this part.

7. As soon as you let go the body, start the timer and begin to count the periods*.

8. Count until the pendulum complete its 10th period. Stop the timer at the end of the 10th period. Write the time measurement on Table (1).

9. Repeat the steps 5-8 three times. DO NOT change anything in experimental setup.

10. Repeat the steps 4-8 with the 2nd body.

*A period is the time spent in one-whole swing. In other words, you should count the time that the mass goes all the way furthest and comes back at the point that you've let it go as a period. Check Figure (2).

2.1.2 Part B: Effect of amplitude difference

In this part, the rope length and the mass will remain same. We'll only change the amplitude. What is your expectation?

1. Measure the mass value of the pendulum body and write it in Eqn (24).
2. There is a long rope with several knots, hanged to the ceiling in our lab. Select 1 of these knots and measure its length. Write the length value in Eqn (23).
3. Hang the body at the knot.
4. Select two different amplitudes, measure them and write them down on Table (2).
5. Pull the body to the 1st amplitude.
6. Make your timer ready.
7. As soon as you let go the body start the timer and begin to count the periods.
8. Count until the pendulum complete its 10th period. Stop the timer at the end of the 10th period. Write the time measurement on Table (2).
9. Repeat the steps 6-8 three times. DO NOT change anything in experimental setup.
10. Repeat the steps 5-8 with the 2nd amplitude.

2.1.3 Part C: Effect of rope length

In this part, the mass and the amplitude will remain same. We'll only change the rope length. What is your expectation?

1. Measure the mass value of the pendulum body and write it in Eqn (26).
2. Choose an amplitude and measure it. Write this value in Eqn (25).
3. There is a long rope with several knots, hanged to the ceiling in our lab. Select 5 of these knots, mark them and measure their length. Write the length values in Table (3). Be careful about that the differences between the length of the knots you choosed should be big enough and nearly same.
4. Hang the body at the shortest of the knots.
5. Make your timer ready.
6. Pull the body a little bit further from the equilibrium. The amount you've pulled should be the same through this part.
7. As soon as you let go the body start the timer and begin to count the periods.
8. Count until the pendulum complete its 10th period. Stop the timer at the end of the 10th period. Write the time measurement on Table (3).
9. Repeat the steps 4-8 three times. DO NOT change anything in experimental setup.
10. Repeat the steps 4-8 with the other knots going from the shortest to the longest.

2.2 Analysis Procedure

- Take a quick look on your tables. They all contains the data of the same measurement third times. Calculate the mean value of these three repetitions and write the results in the relevant cells in tables.
- You have measured the $10T$ in the measurements just so the error would decrease. Divide these values by 10 to get $1T$, *i.e.* the period. Write them down in relevant cells on the tables. These values are your experimental period values.
- By using the formula given in Eqn (18), calculate the theoretical period values.
- By using the percentage error formula given as,

$$P.E. = \% \frac{|T_{the} - T_{exp}|}{T_{the}} \times 100, \tag{19}$$

  calculate and note the percentage errors in the period values.
- Only for the experimental period values in Table (3), calculate the square of them and fill the last column in the Table (3).

Instructions for the graph
You can easily see from Eqn (18) that the plot of L-T is not linear. To linearize it we may do the same old trick, as follows:

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad \Rightarrow \qquad T^2 = 4\pi^2\frac{L}{g}. \tag{20}$$

Now you can see that the plot of L-T^2 is linear and its slope yields $4\pi^2/g$.

1. Bu using the T_{exp} and L values in Table (3), plot the L vs. T^2_{exp} graph.
2. Calculate its slope. This value should be equal to $4\pi^2/g$
3. From slope, obtain the gravitational constant, g. This is your experimental value for g.
4. By using the percentage error formula, calculate the percentage error in the gravitational constant, g.

3 Data & Analysis

3.1 Part A: Effect of mass difference

- Length of rope:

$$L = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ cm. \tag{21}$$

- Amplitude:

$$s_0 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ cm. \tag{22}$$

n	m (gr)	Measurement Repeats	$10T$ $(sec.)$	T_{exp} $(sec.)$	T_{the} $(sec.)$	P.E.
1		$(10T)_1$ $(10T)_2$ $(10T)_3$				%
2		$(10T)_1$ $(10T)_2$ $(10T)_3$				%

Table 1: Data of Part A

3.2 Part B: Effect of amplitude difference

- Length of rope:

$$L = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ cm. \tag{23}$$

- Mass of body:

$$m = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ gr. \tag{24}$$

n	s_0 (cm)	Measurement Repeats	$10T$ $(sec.)$	T_{exp} $(sec.)$	T_{the} $(sec.)$	P.E.
3		$(10T)_1$ $(10T)_2$ $(10T)_3$				%
4		$(10T)_1$ $(10T)_2$ $(10T)_3$				%

Table 2: Data of Part B

3.3 Part C: Effect of rope length

- Amplitude:

$$s_0 = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} cm. \tag{25}$$

- Mass of body:

$$m = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} gr. \tag{26}$$

n	L (cm)	Measurement Repeats	$10T$ ($sec.$)	T_{exp} ($sec.$)	T_{the} ($sec.$)	P.E.	T_{exp}^2 (sec^2)
5		$(10T)_1$ $(10T)_2$ $(10T)_3$				%	
6		$(10T)_1$ $(10T)_2$ $(10T)_3$				%	
7		$(10T)_1$ $(10T)_2$ $(10T)_3$				%	
8		$(10T)_1$ $(10T)_2$ $(10T)_3$				%	
9		$(10T)_1$ $(10T)_2$ $(10T)_3$				%	

Table 3: Data of Part C

4 Conclusions

5 Notes

References

[1] MalVin A Helmholz A Carl Moyer Burton J. Knight. Walter D, Ruderman. *Berkeley Physics Course v.1 Mechanics*. McGraw-Hili, Inc., 1973.

[2] Wikipedia contributors. Polar coordinate system — Wikipedia, the free encyclopedia. `https://en.wikipedia.org/w/index.php?title=Polar_coordinate_system&oldid=870246913`, 2018. [Online; accessed 23-November-2018].

[3] Wikipedia contributors. Small-angle approximation — Wikipedia, the free encyclopedia. `https://en.wikipedia.org/w/index.php?title=Small-angle_approximation&oldid=868436143`, 2018. [Online; accessed 23-November-2018].