Spring - Mass System

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1 Theoretical Background

1.1 Hooke's Law

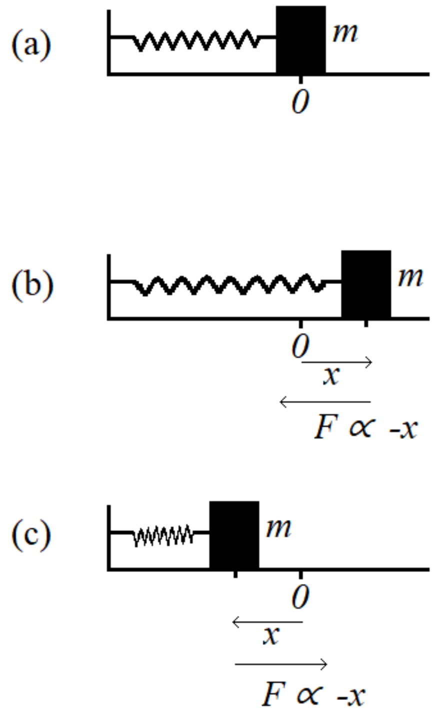

Figure 1: Simple spring-mass system in 1D. One end of the spring is attached to the wall and other end is to the mass. (a) equilibrium status. (b) elongated spring. $\vec{x} = x\hat{\mathbf{i}}$ and $\vec{F} = -kx\hat{\mathbf{i}}$ (c) compressed spring. $\vec{x} = -x\hat{\mathbf{i}}$ and $\vec{F} = kx\hat{\mathbf{i}}$

Let's consider a simple helical spring whose one end attached to a wall. If the spring is released, i.e. there is no compression or elongation then there is no potential energy stored in the spring. Therefore if a mass is attached to the other end of the spring, this mass would not face any force at all. This status is the equilibrium. But if the spring would be compressed or elongated, there would be some amount of potential energy in the spring proportional to the compression/elongation amount. Due to this potential energy, the mass attached to the spring would feel a force. This force always acts as so the potential energy stored in the energy is released, meaning pushes the mass towards the equilibrium point. This force is called as "Hooke's Law" and expressed as,

$$\vec{F} = -k(\vec{r} - \vec{r_0}), \tag{1}$$

where k, $\vec{r}$ and $\vec{r_0}$ are the spring constant, the position vector of the mass and the position vector of the equilibrium point, respectively. Spring constant is a parameter that characterizes the spring. This scalar depends on the material of the spring. We may choose our coordinate system so that the equilibrium point would be the origin of it. Then we get,

$$\vec{F} = -k\vec{r}. \tag{2}$$

According to Newton's 2nd law of motion for the systems with constant mass, the net external force on the system is proportional to its linear acceleration,

$$\vec{F}_{net,ext} = m\ddot{\vec{r}}^{*}. \tag{3}$$

By using Eqn (2) and Eqn (3) together, we gain the equation of motion of a simple spring mass system given in Figure () as follows,

$$\ddot{\vec{r}} + \frac{k}{m}\vec{r} = 0. \tag{4}$$

*Please remember that the acceleration is 2nd order time derivative of the position.

If this mass is moving in only 1 direction, then the motion is 1D. Thus,

$$\ddot{x} + \frac{k}{m}x = 0. \tag{5}$$

Eqn (5) is a 2nd order differential equation with constant coefficients. We may propose a solution such as,

$$x(t) = A\ cos\left(\omega t + \phi\right). \tag{6}$$

Here ω is a constant of dimension $Time^{-1}$. The physical meaning of this constant will be discussed in a moment. Now, if we derivate this proposal with respect to time,

$$\ddot{x} = -\omega^2 A\ cos\left(\omega t + \phi\right), \tag{7}$$

and substitute them into Eqn (5), we get,

$$\begin{aligned} \underbrace{\ddot{x} + \frac{k}{m}x}_{\Downarrow} &= 0, \\ -\omega^2 A\ cos\left(\omega t + \phi\right) + \frac{k}{m} A\ cos\left(\omega t + \phi\right) &= 0, \\ \left(-\omega^2 + \frac{k}{m}\right) A cos\left(\omega t + \phi\right) &= 0. \end{aligned} \tag{8}$$

The general solution of this equation is,

$$\omega = \sqrt{\frac{k}{m}}. \tag{9}$$

This solution means that if ω is equal to the square root of spring constant over mass, then the solution proposed in Eqn (6) is the solution of the differential equation given in Eqn (5). Please remember that, ω is a constant of dimension $1/Time$. In physics, these kind of constants are a type of frequency. This situation hints that ω is the angular frequency of the system and is defined as given in Eqn 9. Therefore we have,

$$f = \frac{\omega}{2\pi}, \tag{10}$$

where f is the frequency of the system. Then it is,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{11}$$

On the other hand, in general the period of a periodical motion is the inverse of its frequency,

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}. \tag{12}$$

1.2 Spring-Mass System in The Influence of Gravity

Let's consider a spring-mass system in vertical direction given in Figure (2). Whe the mass is hanged, the influence of gravity on the mass causes the spring to elongate. Since the mass makes no movement at this position (see Fig (2b&d)), then Newton's 1st law of motion is valid in this system, therefore the net force on the mass vanishes. Then we get,

$$F_{net} = k\Delta l_0 - mg = 0. \tag{13}$$

Note that we may rewrite this formula as,

$$W = k\Delta l_0, \tag{14}$$

where $W = mg$ is the weight of the mass. A simple look at this last formula should make you realize that we may obtain the spring constant from the slope of Δl_0-W plot.

On the other hand, the equilibrium point of the spring-mass system shifts from l_0 to l'. We name this point as "*new equilibrium point*" (see Fig (2b)). Since this position is the new equilibrium, if one shifts the mass from this position, then it would start to oscillate under the influence of unbalanced part of new spring force (see Fig (2c&e)). The total force on this mass is,

$$k(\Delta l_0 + \Delta l) - mg \neq 0. \tag{15}$$

According to Newton's 2nd Law of Motion, an unbalanced force causes an acceleration of the system, which is inversely proportional to its mass. When this statement is formulated, we get;

$$k(\Delta l_0 + \Delta l) - mg \quad = \quad -m\ddot{l} \tag{16}$$

The minus sign in this equation arises from the fact that l tends to decrease, so its time derivative is negative. And please remember the 2nd order time derivative of position is the acceleration. Here, the position of the mass is denoted by l.

$$\begin{aligned} \cancel{k\Delta l_0} + k\Delta l - \cancel{mg} &= -m\ddot{l}, \\ k(l - l') &= -m\ddot{l}. \end{aligned} \tag{17}$$

This equation is a 2nd order differential equation. Let's put this equation in order, as follows;

$$\ddot{l} + \frac{k}{m} l = \underbrace{\frac{k}{m} l'}_{const.} . \tag{18}$$

This equation is similar to the differential equation stated in the last experiment's lab manual other than the constant on the right-hand side. We'll try to find a way to get rid of this extra constant. To this end, we choose to perform following conversion.

$$z \equiv l - l' \qquad \Rightarrow \qquad \ddot{l} = \ddot{z}. \tag{19}$$

Let's organize Eqn (18) and rewrite it in terms of z,

$$\begin{aligned} \ddot{l} \;\; &+ \frac{k}{m}(l - l') = 0 \\ \downarrow \;\; & \quad\; \downarrow \\ \ddot{z} \;\; &+ \frac{k}{m} \quad z \quad\; = 0 \end{aligned} \tag{20}$$

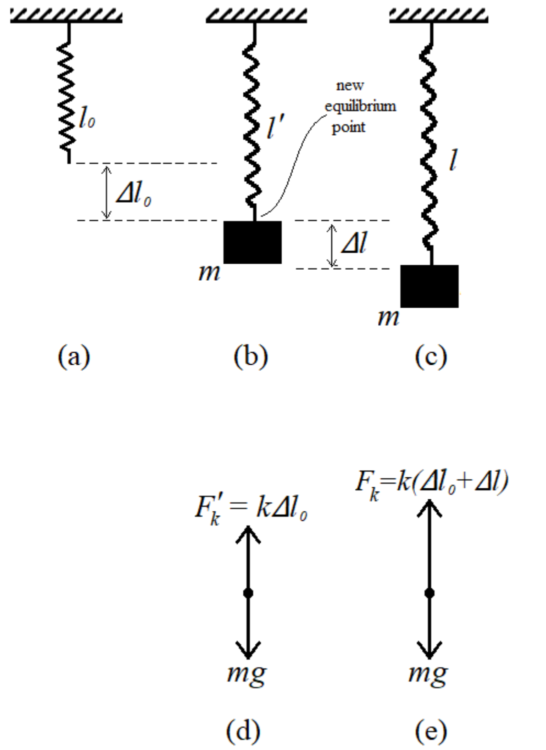

Figure 2: Mass - spring system hanged to the ceiling. (a) spring without mass, l_0 is the relaxed length of the spring by itself. (b) spring with mass, l' is the length of the spring after the mass is hanged. The system is in a state of equilibrium because the weight and the spring force are of equal size and opposite direction. This situation creates a new equilibrium point. And resulting from this equilibrium, l' is constant. (c) spring with mass removed from the new equilibrium point. Δl is the elongation from the new equilibrium point, l is the length of the string in this case. Note that, now the spring force dominates the weight resulting as an unbalanced net force and this triggers the mass to oscillate. Due to this oscillation, l changes over time. (d) Free-body diagram of the mass in Fig (2b). F_k' is spring force, k is spring constant and $\Delta l_0 = l' - l_0$ is the amount of elongation. (e) Free-body diagram of the system in Fig (2c). F_k is new spring force. Under the influence of unbalanced part of this force, the system begins to oscillate.

Please note that in the last line above, we managed to get rid of the extra constant in Eqn (18) and Eqn (20) is the same differential equation from the last experiment. And, naturally, we propose a similar solution,

$$\begin{aligned} z(t) &= A\, cos\,(\omega t + \phi)\,, \\ \ddot{z}(t) &= -\omega^2 A\, cos\,(\omega t + \phi)\,. \end{aligned} \tag{21}$$

Here ω is a constant of dimension $Time^{-1}$ and ϕ is a dimensionless constant. Then we get,

$$\left(-\omega^2 + \frac{k}{m}\right) A\, cos\,(\omega t + \phi) = 0. \tag{22}$$

As we discussed in the lab, the general solution of this equation is,

$$\omega = \sqrt{\frac{k}{m}}. \tag{23}$$

Now let's write the solution for l, as,

$$l(t) = A\, cos\left(\sqrt{\frac{k}{m}}t + \phi\right) + l' \tag{24}$$

Notice that the ω is the angular frequency and ϕ is the phase of this system. Angular frequency measures angular displacement per unit time. Now, we need to apply the initial conditions in order to find A. See from Figure (2c) that the initial position of the mass is $l' + \Delta l$. And, as soon as the mass is released it moves in a way to decrease this initial elongation. In other words, position is in its maximum value at the beginning. Thus the movement begins at the maximum value of the cosine function, *i.e.* the phase is 0. Then,

$$l(0) = A\underbrace{cos\,(\omega 0 + 0)}_{=1} + l' = l' + \Delta l \qquad \Rightarrow \qquad A = \Delta l \tag{25}$$

Δl is the amplitude of this movement and we may rename Δl as L_0 for the sake of simplicity. All in all, the position of the mass in this system is,

$$l(t) = L_0\, cos\left(\sqrt{\frac{k}{m}}t\right) + l', \tag{26}$$

in terms of the time spent during the movement.
We have discussed in the last experiment's lab manual that the frequency is,

$$f = \frac{1}{2\pi}\omega, \tag{27}$$

then the frequency of this system is,

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}, \tag{28}$$

and it's period is,

$$T = 2\pi\sqrt{\frac{m}{k}}. \tag{29}$$

Same in the last experiment, we will investigate the behavior of the period due to the changes in some other physical quantities, such as the mass, the spring constant, etc.

2 Procedure

2.1 Experimental Procedure

Part A: Determination of Spring Constant

1. By using the hook apparatus, hang the spring to the air pipe of the airtable.
2. Using a ruler, measure the relaxed length of the spring and note it in Eqn (34).
3. Hang the mass #1 from Table (1) to the end of the spring by a hanger. With this effect the system will make small oscillations, wait until the system settles down.
4. With a ruler, measure l' and write it in Table (1).
5. Repeat steps 3&4 with other mass values from Table (1).

Part B: Effect of Mass on The Period

1. By using the hook apparatus, hang the spring to the air pipe of the airtable. There should be a sticker attached to the spring. Read the spring constant from it and note this value in Eqn (35).
2. Determine an amount for the amplitude, 1-2cm, and not it in Eqn (36).
3. Hang the mass #1 from Table (2) to the end of the spring by a hanger. With this effect the system will make small oscillations, wait until the system settles down.
4. Set your timer.
5. Pull the mass downwards as the amount of the amplitude you have determined by hand. This amount should be same throughout this part.
6. As soon as you let go the mass, start the timer and begin to count the oscillations.
7. Count until you reach the end of the 10th oscillation and stop your timer at this time.
8. Record your time value in Table (2).
9. Repeat steps 3-8 with other mass values from Table (2).

Part C: Effect of Spring Constant on The Period

1. By using the hook apparatus, hang the spring to the air pipe of the airtable. There should be a sticker attached to the spring. From it, read the spring constant and note this value on Table (3).
2. Hang the mass of 110gr to the end of the spring by a hanger. With this effect the system will make small oscillations, wait until the system settles down.
3. Set your timer.
4. Pull the mass 1-2cm downwards by hand. This amount should be same throughout this part.
5. As soon as you let go the mass, start the timer and begin to count the oscillations.
6. Count until you reach the end of the 10th oscillation and stop your timer at this time.
7. Record your time value in Table (3).
8. Repeat steps 1-7 with three more springs. Select the springs with different spring constants.

Part D: Effect of Amplitude on The Period

1. By using the hook apparatus, hang the spring to the air pipe of the airtable. There should be a sticker attached to the spring. From it, read the spring constant and note this value on Eqn (40).
2. Hang the mass of 110gr to the end of the spring by a hanger. With this effect the system will make small oscillations, wait until the system settles down.
3. Set your timer.
4. Pull the mass downwards by hand as the amount#1 from Table (4).
5. As soon as you let go the mass, start the timer and begin to count the oscillations.
6. Count until you reach the end of the 10th oscillation and stop your timer at this time.
7. Record your time value in Table (4).
8. Repeat steps 3-7 with the second amount for the amplitude from Table (4).

2.2 Analysis Procedure

How to Fill Table (1)

- Calculate the weight of the mass and write it under the column of W on Table (1).
- Bu using the equation,

$$\Delta l_0 = l' - l_0, \tag{30}$$

calculate the elongation in the spring and write in down in Table (1).

How to Fill Table (2-3-4)

- Divide the $10T_{exp}$ values you have measured by 10 and calculate the T_{exp} values. Write them in the relevant cell.
- By using the Eqn (29), calculate the T_{the} and write them in tables.
- By using the percentage error formula,

$$P.E. = \frac{|T_{the} - T_{exp}|}{T_{the}} \times 100, \tag{31}$$

calculate the percentage errors in period values and note them in tables.
- Calculate the square values of the T_{exp} and write them in the relevant cells.
- **Exception:** Only for Table (3), calculate the inverse of spring constants and write them on Table (3).

Plot of Part A

According to the Eqn (14), the slope of the graph of elongation-weight should yield the spring constant.

- Plot Δl_0-W graph.
- Calculate the slope of this graph.
- By using the percentage error formula,

$$P.E. = \% \frac{|k_{the} - k_{exp}|}{k_{the}} \times 100, \tag{32}$$

calculate the percentage error in the spring constant. Take the value from the sticker attached to the spring as the theoretical value.

Plot of Part B

Remember the Eqn (29), period is proportional to the square root of the mass. Thus if you plot the graph of m-T, it will not be linear. So we need the same old trick,

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad \Rightarrow \qquad T^2 = 4\pi^2\frac{m}{k}. \tag{33}$$

It is clear from the equation above that the plot of m-T^2 will be linear and the slope of this graph should yield $4\pi^2/k$.

- Plot m-T^2 graph.
- Calculate the slope of this graph. By using this value evaluate the experimental value for spring constant.
- By using Eqn (32), calculate the percentage error in the spring constant. Take the value from the sticker attached to the spring as the theoretical value.

Plot of Part C

Same in previous part, graph of k-T is not linear. You can see from Eqn (33) that we may plot k^{-1}-T^2 graph which is linear. It is clear from the Eqn (33) the slope of this graph should yield $4\pi^2 m$.

- Plot k^{-1}-T^2 graph.
- Calculate the slope of this graph. By using this value evaluate the experimental value for the mass.
- By using Eqn (32), calculate the percentage error in the mass.

3 Data & Analysis

Part A: Determination of Spring Constant

- Relaxed length of spring

$$l_0 = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} cm. \tag{34}$$

#	m (gr)	W (dyn)	l' (cm)	Δl_0 (cm)
1	110			
2	120			
3	130			
4	140			
5	150			

Table 1: Data for determination of spring constant

Part B: Effect of Mass on The Period

- Spring constant:

$$k = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ dyn\, cm^{-1}. \quad (35)$$

- Amplitude:

$$\Delta l = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ cm. \quad (36)$$

#	m (gr)	$10T_{exp}$ (sec)	T_{exp} (sec)	T_{the} (sec)	P.E.	T^2_{exp} (sec^2)
6	110				%	
7	120				%	
8	130				%	
9	140				%	
10	150				%	

Table 2: Data for Effect of Mass on The Period

Part C: Effect of Spring Constant on The Period

- Mass of the body:

$$m = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ gr. \quad (37)$$

- Amplitude:

$$\Delta l = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ cm. \quad (38)$$

#	k $(dyn\,cm^{-1})$	k^{-1} $(dyn^{-1}\,cm)$	$10T_{exp}$ (sec)	T_{exp} (sec)	T_{the} (sec)	P.E.	T^2_{exp} (sec^2)
11						%	
12						%	
13						%	
14						%	

Table 3: Data for Effect of Spring Constant on The Period

Part D: Effect of Amplitude on The Period

- Mass of the body:

$$m = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ gr. \quad (39)$$

- Spring constant:

$$k = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ dyn\,cm^{-1}. \quad (40)$$

#	Δl (cm)	$10T_{exp}$ (sec)	T_{exp} (sec)	T_{the} (sec)	P.E.	T^2_{exp} (sec^2)
15	2				%	
16	4				%	

Table 4: Data for Effect of Amplitude on The Period

References